# Lecture Notes on Interest Rate Models

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# Feynman-Kac Formula

- Consider random variable $G(X_T)$ from solution of SDE
$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t.$$

- Let $g(t, x) = E^{t,x}[G(X_T)]$. That is, we start the SDE at time $t$ from state $x$, $g(t, x)$ denotes the expectation of generated sample of $G(X_T)$.

- Can show that a.s. $E[G(X_T)|\mathcal{F}_t] = g(t, X_t)$.

- Since $g(t, X_t)$ is a martingale, Itô's Lemma provides the p.d.e. satisfied by $g(t, X_t)$. Hiding arguments $(t, X_t)$,
$$dg = g_t dt + g_x dX_t + \frac{1}{2} g_{xx} dX_t.dX_t$$

# Feynman-Kac Formula

- Setting the $dt$ term to zero for each $t$ and $X_t$, we get the pde

$$g_t(t,x) + \mu(t,x)g_x(t,x) + \frac{1}{2}\sigma^2(t,x)g_{xx}(t,x) = 0$$

- with the boundary condition $g(T,x) = G(x)$ for all $x$.
- In options pricing we often consider option price

$$c(t,x) = E^{t,x}(e^{r(T-t)}G(X_T)).$$

In this case, $e^{rt}c(t,X_t)$ is a martingale, and one can similarly identify the Feynman-Kac formula or the pde associated with $c(t,x)$.

# Options on Geometric Brownian motion

- Risky security under physical probability measure

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

- Under risk neutral measure

$$dS_t = rS_t dt + \sigma S_t d\tilde{B}_t.$$

- For option with payoff $G(S_T)$, the option price

$$c(t, S_t) = \tilde{E}[e^{-r(T-t)} G(S_T) | \mathcal{F}_t]$$

- $\Rightarrow e^{-rt} c(t, S_t)$ is a martingale.

- Expand using Ito's Lemma. Find the *dt* term and set it to zero.

$$d(e^{-rt}c) = -re^{-rt}cdt + e^{-rt}dc$$

# Options on Geometric Brownian motion

- $$= -re^{-rt} c dt + e^{-rt} \left[ c_t dt + c_s dS_t + \frac{1}{2} c_{ss} dS_t . dS_t \right]$$
  $$= -re^{-rt} c dt + e^{-rt} \left[ c_t dt + c_s (r S_t dt + \sigma S_t d\tilde{B}_t) + \frac{1}{2} c_{ss} \sigma^2 S_t^2 dt \right]$$
  $$= e^{-rt} \left[ -rc + c_t + r S_t c_s + \frac{1}{2} \sigma^2 S_t^2 c_{ss} \right] dt + e^{-rt} \sigma S_t c_s d\tilde{B}_t .$$

- Setting $dt$ term to 0, for all $t$ and $S_t$, we get the Black Scholes equation

  $$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x)$$

  and

  $$c(T,x) = G(x) \ \forall x \geq 0$$

  .

# Interest Rate Models

- Assume that short rate follows the process

$$dr_t = \beta(t, r_t)dt + \sigma(t, r_t)d\tilde{B}_t$$

  under the risk neutral measure (which is assumed to exist).

- This is a typical procedure
  - Assume risk neutral measure $\tilde{P}$ exists and has a parametric form
  - Estimate parameters from existing Vanilla "liquid" options or securities.
  - Price other more bespoke or exotic options using this model.

# Zero Coupon Bond

- Consider the discount rate process

$$R_t = exp\left[-\int_0^t r_s ds\right], \quad dR_t = -r_t R_t dt$$

- A zero coupon bond promises to pay a face amt. (w.l.o.g. = Rs. 1) at fixed maturity date $T$.

- Let $B(t, T)$ denote the price of the bond at time $t < T$

- Under $\tilde{P}$, $R_t B(t, T)$ is a martingale. Since $B(T, T) = 1$,

$$R_t B(t, T) = \tilde{E}(R_T | \mathcal{F}_t)$$

- $$\Rightarrow B(t,T) = \tilde{E}\left(e^{-\int_t^T r_s ds} | \mathcal{F}_t\right)$$

- Since $dr_t = \beta(t, r_t)dt + \sigma(t, r_t)d\tilde{B}_t$ is a Markov process,

- $B(t,T)$ depends only on $t$, $r_t$.

- We have $B(t,T) = f(t, r_t)$ for some $f$.

# P.D.E. satisfied by the bond price I

- Applying Itô's lemma to $R_t f(t, r_t)$ we get

$$d(R_t f(t, r_t)) = -r_t R_t f \, dt + R_t[f_t dt + f_r dR_t + \frac{1}{2} f_{rr} dR_t dR_t]$$

$$= R_t[-r_t f + f_t + \beta f_r + \frac{1}{2}\sigma^2 f_{rr}]dt + R_t \sigma f_r d\tilde{B}_t$$

- Since $R_t B(t, T)$ is a martingale, it is a stochastic integral, the $dt$ term is zero, we obtain the p.d.e.

$$f_t(t, r) + \beta(t, r) f_r(t, r) + \frac{1}{2}\sigma^2(t, r) f_{rr}(t, r) = r f(t, r)$$

- with terminal condition $f(T, r) = 1$ for all r.

# Hull White Interest Rate Model

- Consider the short rate process

$$dr_t = (a_t - b_t r_t)dt + \sigma_t d\tilde{B}_t$$

where $a_t$, $b_t$ and $\sigma_t$ are deterministic functions of $t$.

- The associated bond price pde for each $t$ and $r$ is

$$f_t + (a_t - b_t r)f_r + \frac{1}{2}\sigma_t^2 f_{rr} = rf$$

with $f(T, r) = 1$ for all $r$.

- Suppose that

$$f(t, r) = e^{-rC(t,T) - A(t,T)}$$

for some deterministic functions $C(t, T)$ and $A(t, T)$.

- The bond yield $Y(t, T)$ given by

$$B(t, T) = e^{-Y(t,T)(T-t)}.$$

- Above implies that

$$Y(t, T) = -\frac{1}{T-t}(\ln f(t, r)) = \frac{1}{T-t}(rC(t, T) + A(t, T))$$

is an affine function of $r$.

# Solving Hull White Interest Rate Model

- Then,

$$f_t(t,r) = (-rC'(t,T) - A'(t,T))f(t,r)$$

- and $f_r = -C(t,T)f$, and $f_{rr} = C^2(t,T)f$
- The PDE for each $r$ becomes

$$[(-C'(t,T) + b_t C(t,T) - 1)r - A'(t,T)$$
$$-a_t C(t,T) + \frac{1}{2}\sigma_t^2 C^2(t,T)]f(t,r) = 0$$

- Implies that $C'(t,T) = b_t C(t,T) - 1$ and

$$A'(t,T) = -a_t C(t,T) + \frac{1}{2}\sigma_t^2 C^2(t,T)$$

- with terminal conditions $C(T,T) = 0 = A(T,T)$.

# Hull White Interest Rate Model

- Can check that

$$C(t, T) = \int_t^T e^{-\int_t^s b_r dr} ds$$

- and

$$A(t, T) = \int_t^T \left( a_s C(s, T) - \frac{1}{2} \sigma_s^2 C^2(s, T) \right) ds$$

and

$$B(t, T) = e^{-r_t C(t,T) - A(t,T)}$$

for $0 \leq t \leq T$.

# CIR Model

Consider CIR model

$$dr_t = (a - br_t)dt + \sigma\sqrt{r_t}d\tilde{B}_t$$

Then, again $B(t, T) = f(t, r_t)$ where for each $t$ and $r$,

$$f_t + (a - br)f_r + \frac{1}{2}\sigma^2 r f_{rr} = rf.$$

Again one can solve by assuming affine form

$$f(t, r) = e^{-rC(t,T) - A(t,T)}.$$

# Pricing call option on a zero coupon bond

- Call on a bond option $0 \leq t \leq T_1 \leq T_2$
  - $T_2$: maturity date for zero coupon bond
  - $T_1 < T_2$: Expiration date for European Call on this bond
  - We want to find value of the option at time $t$.
- Let $f(t, r_t) = B(t, T_2)$ where $f(t, r_t)$ is the bond price.
- Call option price

$$C(t, r_t) = \tilde{E}\left[ e^{-\int_t^{T_1} r_s ds} (f(T_1, r_{T_1}) - K)^+ \middle| \mathcal{F}_t \right]$$

- Now $R_t C(t, r_t)$ is a martingale.
- Write its Itô representation and set the *dt* term to zero.
- Boundary condition is $C(T_1, r) = (f(T_1, r) - K)^+$ for every $r$.
- Setting the *dt* term in $d(R_t C(t, r_t))$ expansion to zero, we get

$$C_t + \beta C_r + \frac{1}{2}\sigma^2 C_{rr} = rC$$

where recall that $dr_t = \beta(t, r_t)dt + \sigma(t, r_t)d\tilde{B}_t$.